

# The Color Gauge Invariance and a Possible Origin of Mass in QCD

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**Abstract** The general scale parameter, having the dimensions of mass squared, is dynamically generated in the QCD gluon sector. It is introduced through the difference between the regularized full gluon self-energy and its value at some finite point. It violates transversality of the full gluon self-energy. The Slavnov-Taylor identity for the full gluon propagator, when it is given by the corresponding equation of motion, is also violated by it. So in order to maintain both transversality and the identity it should be disregarded from the very beginning, i.e., put formally zero everywhere. However, we have shown how to preserve the above-mentioned identity at non-zero mass squared parameter. This allows one to establish the structure of the full gluon propagator when it is explicitly present. Its contribution does not survive in the perturbation theory regime, when the gluon momentum goes to infinity. At the same time, its contribution dominates the structure of the full gluon propagator when the gluon momentum goes to zero. We have also proposed a method how to restore transversality of the relevant gluon propagator in a gauge invariant way, while keeping the mass squared parameter “alive”.

**Keywords** Quantum chromodynamics · Gauge invariance/symmetry · Transversality · Identity · Gluon propagator · Equation of motion · Gluon self-energy · Subtraction · Mass scale parameter

## 1 Introduction

Quantum Chromodynamics (QCD) [1–4] is widely accepted as a realistic quantum field gauge theory of the strong interactions not only at the fundamental (microscopic) quark-gluon level but at the hadronic (macroscopic) level as well. It is a  $SU(3)$  color gauge invariant theory but:

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- (i) Due to color confinement, the gluon (unlike the photon) is not a physical state. Moreover, there is no physical amplitude to which the gluon self-energy (like the photon self-energy) may directly contribute.
- (ii) In contrast to the conserved currents in Quantum Electrodynamics (QED), the color-conserved currents do not play any role in the extraction of physical information from the  $S$ -matrix elements for the corresponding physical processes and quantities in QCD. In other words, the conserved color currents do not contribute directly to the  $S$ -matrix elements describing this or that physical process/quantity. For this their color-singlet counterparts, which can even be partially conserved, are relevant. For example, an important physical QCD parameter such as the pion decay constant is given by the following  $S$ -matrix element:  $\langle 0|J_{5\mu}^i(0)|\pi^j(q)\rangle = iq_\mu F_\pi \delta^{ij}$ , where  $J_{5\mu}^i(0)$  is the axial-vector current, while  $|\pi^j(q)\rangle$  describes the pion bound-state amplitude, and  $i, j$  are flavor indices.
- (iii) In QCD (contrary to QED) there exists direct evidence/indication that transversality of the full gluon self-energy, as well as the Slavnov-Taylor (ST) identity for the full gluon propagator, as it is determined by the corresponding equation of motion, is violated. Indeed, there is no regularization scheme (preserving or not gauge invariance) in which the transversality condition and the ST identity could be satisfied unless the so-called constant skeleton tadpole term (or, equivalently, its re-defined counterpart, which we will call the general mass scale parameter in what follows) is to be disregarded from the very beginning, i.e., put formally zero everywhere.

However, our main goal in this paper is to show that the tadpole term is consistent with the color gauge invariance in QCD, i.e., it is maintained at non-zero general mass scale parameter as well.

## 2 The Full Gluon Self-Energy

For our purpose it is convenient to begin with the general description of the Schwinger-Dyson (SD) equation for the full gluon propagator. It can be written as follows:

$$D_{\mu\nu}(q) = D_{\mu\nu}^0(q) + D_{\mu\rho}^0(q)i\Pi_{\rho\sigma}(q; D)D_{\sigma\nu}(q), \tag{2.1}$$

where

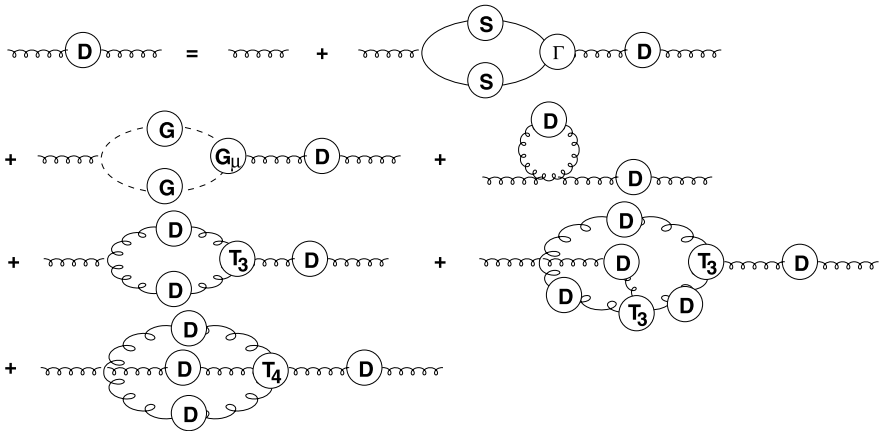
$$D_{\mu\nu}^0(q) = i \left\{ T_{\mu\nu}(q) + \xi L_{\mu\nu}(q) \right\} \frac{1}{q^2} \tag{2.2}$$

is the free gluon propagator, and  $\xi$  is the gauge-fixing parameter. Also, here and everywhere below  $T_{\mu\nu}(q) = \delta_{\mu\nu} - (q_\mu q_\nu / q^2) = \delta_{\mu\nu} - L_{\mu\nu}(q)$ , as usual in Euclidean metrics (see remarks below as well).  $\Pi_{\rho\sigma}(q; D)$  is the full gluon self-energy which depends on the full gluon propagator due to the non-abelian character of QCD. Thus the gluon SD equation is highly nonlinear (NL). Evidently, we omit the color group indices, since for the gluon propagator (and hence for its self-energy) they factorize, for example  $D_{\mu\nu}^{ab}(q) = D_{\mu\nu}(q)\delta^{ab}$ .

The full gluon self-energy  $\Pi_{\rho\sigma}(q; D)$  is the sum of a few terms (see Fig. 1),

$$\Pi_{\rho\sigma}(q; D) = \Pi_{\rho\sigma}^q(q) + \Pi_{\rho\sigma}^{gh}(q) + \Pi_{\rho\sigma}^t(D) + \Pi_{\rho\sigma}^{(1)}(q; D^2) + \Pi_{\rho\sigma}^{(2)}(q; D^4) + \Pi_{\rho\sigma}^{(2')}(q; D^3), \tag{2.3}$$

where  $\Pi_{\rho\sigma}^q(q)$  describes the skeleton loop contribution due to the quark degrees of freedom (it is an analog of the vacuum polarization tensor in QED), while  $\Pi_{\rho\sigma}^{gh}(q)$  describes



**Fig. 1** The SD equation for the full gluon propagator

the skeleton loop contribution associated with the ghost degrees of freedom. Since neither of the skeleton loop integrals depends on the full gluon propagator  $D$ , they represent the linear contribution to the gluon SD equation.  $\Pi_{\rho\sigma}^t(D)$  is the so-called constant skeleton tadpole term.  $\Pi_{\rho\sigma}^{(1)}(q; D^2)$  represents the skeleton loop contribution, which contains the triple gluon vertices only.  $\Pi_{\rho\sigma}^{(2)}(q; D^4)$  and  $\Pi_{\rho\sigma}^{(2')} (q; D^3)$  describe topologically independent skeleton two-loop contributions, which combine the triple and quartic gluon vertices. All these quantities are given by the corresponding loop diagrams in Fig. 1. The last four terms explicitly contain the full gluon propagators in the corresponding powers symbolically shown above. They thus form the NL part of the gluon SD equation. The analytical expressions for the corresponding skeleton loop integrals [5] (in which the symmetry coefficients and signs have been included, for convenience) are of no importance here, since we are not going to introduce into them any truncation/approximation or choose some special gauge. Let us note in advance that here and below the signature is Euclidean, since it implies  $q_i \rightarrow 0$  when  $q^2 \rightarrow 0$  and vice-versa. All the quantities which contribute to the full gluon self-energy (2.3) are tensors, having the dimensions of mass squared. All these skeleton loop integrals are therefore quadratically divergent in perturbation theory (PT), and so they are assumed to be regularized, as discussed below.

### 3 The Subtractions

Let us subtract from the full gluon self-energy (2.3) its value at  $q = 0$ . Thus, one obtains

$$\Pi_{\rho\sigma}^s(q; D) = \Pi_{\rho\sigma}(q; D) - \Pi_{\rho\sigma}(0; D) = \Pi_{\rho\sigma}(q; D) - \delta_{\rho\sigma} \Delta^2(D), \quad (3.1)$$

which is nothing but the definition of the subtracted full gluon self-energy  $\Pi_{\rho\sigma}^s(q; D)$ . Contrary to QED, QCD being a non-abelian gauge theory can suffer from infrared (IR) singularities in the  $q^2 \rightarrow 0$  limit due to the self-interaction of massless gluon modes. Thus the initial subtraction at zero in the definition (3.1) may be dangerous [1]. That is why in all the quantities below the dependence on the finite (slightly different from zero) dimensionless subtraction point  $\alpha$  is to be understood. In other words, all the subtractions at zero and the

Taylor expansions around zero should be understood as the subtractions at  $\alpha$  and the Taylor expansions near  $\alpha$ , where they are justified to be used. From a technical point of view, however, it is convenient to put formally  $\alpha = 0$  in all the derivations below, and to restore the explicit dependence on non-zero  $\alpha$  in all the quantities only at the final stage. At the same time, in all the quantities where the dependence on  $\lambda$  (which is the dimensionless ultraviolet (UV) regulating parameter) and  $\alpha$  is not shown explicitly, nevertheless, it should be assumed. For example,  $\Delta^2(D) \equiv \Delta^2(\lambda, \alpha; D)$  and similarly for all other quantities (evidently, in all the subtracted quantities like this one  $D$  depends on the loop(s) variable(s) only). This means that all the expressions are regularized (i.e., they become finite), and thus a mathematical meaning is assigned to all of them. For our purpose, in principle, it is not important how  $\lambda$  and  $\alpha$  have been introduced. They should be removed at the final stage only as a result of the self-consistent renormalization program.

From the subtraction (3.1) it follows that the general scale parameter  $\Delta^2(D)$ , having the dimensions of mass squared, is dynamically generated in the QCD gluon sector. It is defined as the value of the regularized full gluon self-energy at some finite point (see discussion above). It is mainly due to the nonlinear interaction of massless gluon modes plus the linear contributions from quark and ghost degrees of freedom, namely

$$\Delta^2(D) = \Pi_t(D) + \Pi_q(0) + \Pi_g(0; D) = \Delta_t^2(D) + \Delta_q^2 + \Delta_g^2(D), \tag{3.2}$$

where

$$\Delta_g^2(D) \equiv \Pi_g(0; D) = \sum_a \Pi_a(0; D) = \sum_a \Delta_a^2(D), \tag{3.3}$$

and the index “ $a$ ” runs as follows:  $a = gh, (1), (2), (2')$ . The tensor indices have been omitted, so in this case all the indices  $t, q, g$ , and hence  $a$ , are subscripts. In these relations all the quadratically divergent constants  $\Pi_t(D) \equiv \Delta_t^2(D)$ ,  $\Pi_q(0) \equiv \Delta_q^2$ , and  $\Pi_a(0; D) \equiv \Delta_a^2(D)$ , having the dimensions of mass squared, are given by the corresponding regularized skeleton loop integrals at  $q^2 = 0$  that appear in (2.3). Let us remind that no truncations/approximations/assumptions, and no special gauge choice are made in the above-mentioned loop integrals, which contribute to the regularized full gluon self-energy.

The subtracted gluon self-energy (3.1)

$$\Pi_{\rho\sigma}^s(q; D) \equiv \Pi^s(q; D) = \Pi_q^s(q) + \Pi_g^s(q; D) = \Pi_q^s(q) + \sum_a \Pi_a^s(q; D) \tag{3.4}$$

is free of the tadpole contribution, because  $\Pi_t^s(D) = \Pi_t(D) - \Pi_t(D) = 0$ , by definition, at any  $D$ , while in the gluon self-energy (2.3) it is explicitly present.

### 4 Transversality of the Full Gluon Self-Energy

Contracting the full gluon self-energy (2.3) with  $q_\rho$ , it can be reduced to the two independent transversality conditions, namely

$$q_\rho \Pi_{\rho\sigma}(q; D) = q_\rho \Pi_{\rho\sigma}^q(q) + q_\rho \Pi_{\rho\sigma}^g(q; D), \tag{4.1}$$

where the pure gluon contribution is defined as follows:

$$\Pi_{\rho\sigma}^g(q; D) = \Pi_{\rho\sigma}^t(D) + \Pi_{\rho\sigma}^{gh}(q) + \Pi_{\rho\sigma}^{(1)}(q; D^2) + \Pi_{\rho\sigma}^{(2)}(q; D^4) + \Pi_{\rho\sigma}^{(2')}(q; D^3). \tag{4.2}$$

### 4.1 The Quark Contribution

The color currents conservation condition (quite similar to the current conservation in QED) implies

$$q_\rho \Pi_{\rho\sigma}^q(q) = q_\sigma \Pi_{\rho\sigma}^q(q) = 0. \tag{4.3}$$

Let us show that then the corresponding constant skeleton quark loop contribution  $\Delta_q^2$  to (3.2) has to be discarded from the very beginning. For this purpose, it is instructive to show the subtraction of the quark part explicitly as follows:

$$\Pi_{\rho\sigma}^{s(q)}(q) = \Pi_{\rho\sigma}^q(q) - \Pi_{\rho\sigma}^q(0) = \Pi_{\rho\sigma}^q(q) - \delta_{\rho\sigma} \Delta_q^2. \tag{4.4}$$

At the same time, the general decompositions of the quark part and its subtracted counterpart into the independent tensor structures are

$$\begin{aligned} \Pi_{\rho\sigma}^q(q) &= T_{\rho\sigma}(q)q^2\Pi(q^2) + q_\rho q_\sigma \tilde{\Pi}(q^2), \\ \Pi_{\rho\sigma}^{s(q)}(q) &= T_{\rho\sigma}(q)q^2\Pi^s(q^2) + q_\rho q_\sigma \tilde{\Pi}^s(q^2). \end{aligned} \tag{4.5}$$

Here and everywhere below all the invariant functions are dimensionless ones of their argument  $q^2$ . In addition both invariant functions  $\Pi^s(q^2)$  and  $\tilde{\Pi}^s(q^2)$  cannot have the power-type singularities (or, equivalently, the pole-type ones) at small  $q^2$ , since  $\Pi_{\rho\sigma}^{s(q)}(0) = 0$ , by definition; see (4.4); otherwise they remain arbitrary. Let us also note that from the color currents conservation condition (4.3) it follows that  $\tilde{\Pi}(q^2) = 0$ , so  $\Pi_{\rho\sigma}^q(q)$  is purely transversal, indeed.

On the other hand, contracting the relation (4.4) with  $q_\rho$ , on account of the relation (4.3) and the second of the relations (4.5), one obtains

$$\tilde{\Pi}^s(q^2) = -\frac{\Delta_q^2}{q^2}. \tag{4.6}$$

This, however, is impossible since  $\tilde{\Pi}^s(q^2)$  cannot have the power-type singularities at small  $q^2$ , as underlined above. The only solution to the previous relation is to disregard  $\Delta_q^2$  from the very beginning, i.e., put formally zero, and hence  $\tilde{\Pi}^s(q^2) = 0$  as well. Thus, one has

$$\Delta_q^2 = 0, \tag{4.7}$$

so the subtracted quark part  $\Pi_{\rho\sigma}^{s(q)}(q)$  becomes also transversal  $q_\rho \Pi_{\rho\sigma}^{s(q)}(q) = 0$ , and  $\Pi_{\rho\sigma}^q(q) = \Pi_{\rho\sigma}^{s(q)}(q)$ , which yields  $\Pi(q^2) = \Pi^s(q^2)$ . This describes a general situation, when just the initial transversality condition (4.3) for  $\Pi_{\rho\sigma}^q(q)$  lowers the quadratic divergence of the corresponding loop integral(s) to a logarithmic one at large  $q^2$ . They may still present in its subtracted counterpart  $\Pi_{\rho\sigma}^{s(q)}(q)$ . This is in complete analogy with QED, since there only electron-positron skeleton loop (the vacuum polarization tensor) contributes to the full photon self-energy.

Let us underline that in obtaining this result no any regularization scheme (preserving or not gauge invariance) has been used. It has been also obtained without using PT (all the full Green's functions which are present in the quark skeleton loop integral have not been replaced by their free PT counterparts). No special gauge choice has been made as well.

### 4.2 The Pure Gluon Contribution

In the same way, however,

$$q_\rho \Pi_{\rho\sigma}^g(q; D) = q_\rho \left[ \Pi_{\rho\sigma}^t(D) + \Pi_{\rho\sigma}^{gh}(q) + \Pi_{\rho\sigma}^{(1)}(q; D^2) + \Pi_{\rho\sigma}^{(2)}(q; D^4) + \Pi_{\rho\sigma}^{(2')}(q; D^3) \right] \neq 0, \tag{4.8}$$

and hence  $q_\rho \Pi_{\rho\sigma}(q; D) \neq 0$ , unless the constant skeleton tadpole term  $\Pi_{\rho\sigma}^t(D)$  is discarded from the very beginning. So omitting it in the relation (4.8), one obtains

$$q_\rho \left[ \Pi_{\rho\sigma}^{gh}(q) + \Pi_{\rho\sigma}^{(1)}(q; D^2) + \Pi_{\rho\sigma}^{(2)}(q; D^4) + \Pi_{\rho\sigma}^{(2')}(q; D^3) \right] = 0. \tag{4.9}$$

It should be noted that none of these quantities can satisfy the corresponding transversality condition separately from each other, i.e, similarly to the relation (4.3). The role of ghost degrees of freedom is to cancel the unphysical (longitudinal) component of the full gluon propagator. Therefore the transversality condition (4.9) is important for ghosts to fulfill their role, and thus to maintain unitarity of the  $S$ -matrix in QCD.

On account of the above-mentioned relation (4.7), the general scale parameter (3.2) becomes

$$\Delta^2(D) = \Delta_t^2(D) + \Delta_g^2(D). \tag{4.10}$$

It is worth emphasizing that it does not matter whether we will discard  $\Delta_g^2(D)$  from the very beginning, in accordance with the transversality condition (4.9) (in complete analogy with the previous quark case), or not. In the explicit presence of the tadpole term  $\Delta_t^2(D)$  this is not important. Not losing generality, it can be included into the tadpole term itself, and the general mass scale parameter  $\Delta^2(D)$  in the relation (4.10) is to be considered as the re-defined tadpole term.

Let us continue with the general decompositions of the full gluon self-energy and its subtracted counterpart, which enter the subtraction (3.1), as follows:

$$\begin{aligned} \Pi_{\rho\sigma}(q; D) &= T_{\rho\sigma}(q)q^2\Pi(q^2; D) + q_\rho q_\sigma \tilde{\Pi}(q^2; D), \\ \Pi_{\rho\sigma}^s(q; D) &= T_{\rho\sigma}(q)q^2\Pi^s(q^2; D) + q_\rho q_\sigma \tilde{\Pi}^s(q^2; D), \end{aligned} \tag{4.11}$$

where again all the invariant functions of  $q^2$  are dimensionless ones, while in addition the invariant functions  $\Pi^s(q^2; D)$  and  $\tilde{\Pi}^s(q^2; D)$  cannot have the pole-type singularities in the  $q^2 \rightarrow 0$  limit, since  $\Pi_{\rho\sigma}^s(0; D) = 0$ , by definition; otherwise all the invariant functions remain arbitrary. Of course, they are different from those of the relations (4.5).

Contracting them with  $q_\rho$  along with the subtraction (3.1), one obtains

$$\tilde{\Pi}(q^2; D) = \tilde{\Pi}^s(q^2; D) + \frac{\Delta^2(D)}{q^2}, \tag{4.12}$$

and

$$\Pi(q^2; D) = \Pi^s(q^2; D) + \frac{\Delta^2(D)}{q^2}. \tag{4.13}$$

It is worth emphasizing that the full gluon self-energy has a massless single particle singularity due to non-zero mass scale parameter  $\Delta^2(D)$ , which is of the non-perturbative (NP) origin. At the same time, its subtracted counterpart cannot have such a singularity, as

mentioned above. In other words, this means that in the explicit presence of the mass scale parameter both invariant functions of the full gluon self-energy gain additional contributions due to it (of course, not only at some finite subtraction point  $q^2 = \mu^2 \neq 0$ ). If  $\Delta^2(D)$  is welcome in the transversal invariant function  $\Pi(q^2; D)$ , it is not welcome in its longitudinal counterpart  $\tilde{\Pi}(q^2; D)$ , since just it violates the ST identity. Let us note that transversality of the full gluon self-energy together with its subtracted counterpart can be achieved only in the formal  $\Delta^2(D) = 0$  limit (for detail discussion of these preliminary remarks see sections below). So in the general case of non-zero  $\Delta^2(D)$  only two possibilities remain.

(i) Both are not transversal and then

$$\begin{aligned} q_\rho \Pi_{\rho\sigma}(q; D) &= q_\sigma q^2 \tilde{\Pi}(q^2; D) = q_\sigma [q^2 \tilde{\Pi}^s(q^2; D) + \Delta^2(D)] \neq 0, \\ q_\rho \Pi_{\rho\sigma}^s(q; D) &= q_\sigma q^2 \tilde{\Pi}^s(q^2; D) = q_\sigma [q^2 \tilde{\Pi}(q^2; D) - \Delta^2(D)]. \end{aligned} \tag{4.14}$$

The last inequality in the first of the relations (4.14) follows from the fact that  $\tilde{\Pi}^s(q^2; D)$  cannot have a single particle singularity  $-\Delta^2(D)/q^2$  in order to cancel  $\Delta^2(D)$ .

(ii) Transversality of the subtracted gluon self-energy is maintained, i.e.,  $\tilde{\Pi}^s(q^2; D) = 0$  and then

$$q_\rho \Pi_{\rho\sigma}(q; D) = q_\sigma q^2 \tilde{\Pi}(q^2; D) = q_\sigma \Delta^2(D) \neq 0, \quad q_\rho \Pi_{\rho\sigma}^s(q; D) = 0. \tag{4.15}$$

Contrary to the first case, now we know how precisely the transversality of the full gluon self-energy is violated. So it is always violated at non-zero mass scale parameter  $\Delta^2(D)$ . In this connection one thing should be made perfectly clear. It is the initial subtraction (3.1) which leaves the subtracted gluon-self energy logarithmical divergent only, and hence the invariant function  $\Pi^s(q^2; D)$  is free of the quadratic divergences, but a logarithmic ones can be still present in it, at any  $D$ . Since the transversality condition for the full gluon self-energy is violated in these relations, that is why we cannot disregard  $\Delta^2(D)$  from the very beginning (compare with the pure quark case considered above).

### 5 The ST Identity for the Full Gluon Propagator

In order to calculate the physical observables in QCD from first principles, we need the full gluon propagator rather than the full gluon self-energy. The basic relation to which the full gluon propagator should satisfy is the corresponding ST identity

$$q_\mu q_\nu D_{\mu\nu}(q) = i\xi. \tag{5.1}$$

It is a consequence of the color gauge invariance/symmetry of QCD, and therefore “is an exact constraint on any solution to QCD” [1]. This is true for any other ST identities. Being a result of this exact symmetry, it is the general one, and it is important for the renormalization of QCD. If some equation, relation or the regularization scheme, etc. do not satisfy it automatically, i.e., without any additional conditions, then they should be modified and not this identity (identity is an equality, where both sides are the same, i.e., there is no room for additional conditions). In other words, all the relations, equations, regularization schemes, etc. should be adjusted to it and not vice versa. It implies that the general tensor decomposition of the full gluon propagator is

$$D_{\mu\nu}(q) = i \left\{ T_{\mu\nu}(q) d(q^2) + \xi L_{\mu\nu}(q) \right\} \frac{1}{q^2}, \tag{5.2}$$

where the invariant function  $d(q^2)$  is the corresponding Lorentz structure of the full gluon propagator (sometimes we will call it as the full effective charge (“running”), for simplicity). Let us emphasize once more that these basic relations are to be satisfied in any case, for example, whether the tadpole term itself or any other mass scale parameter is put formally zero or not.

On account of the exact relations (4.11), (4.12) and (4.13), the initial gluon SD equation (2.1) can be equivalently re-written down as follows:

$$D_{\mu\nu}(q) = D_{\mu\nu}^0(q) + D_{\mu\rho}^0(q)iT_{\rho\sigma}(q)[q^2\Pi^s(q^2; D) + \Delta^2(D)]D_{\sigma\nu}(q) + D_{\mu\rho}^0(q)iL_{\rho\sigma}(q)q^2\tilde{\Pi}(q^2; D)D_{\sigma\nu}(q). \tag{5.3}$$

Contracting this equation with  $q_\mu$  and  $q_\nu$ , one arrives at

$$q_\mu q_\nu D_{\mu\nu}(q) = i\xi - i\xi^2\tilde{\Pi}(q^2; D) = i\xi \left(1 - \xi\tilde{\Pi}(q^2; D)\right), \tag{5.4}$$

so the ST identity (5.1) is not automatically satisfied. In order to get from this relation the ST identity, one needs to put

$$\tilde{\Pi}(q^2; D) = 0, \tag{5.5}$$

which is equivalent to

$$\tilde{\Pi}^s(q^2; D) = -\frac{\Delta^2(D)}{q^2}, \tag{5.6}$$

as it follows from the relation (4.12). This, however, is impossible since  $\tilde{\Pi}^s(q^2; D)$  cannot have the power-type singularities at small  $q^2$ , as underlined above. The only solution to the previous relation is to disregard  $\Delta^2(D)$  from the very beginning, i.e., put formally zero

$$\Delta^2(D) = 0 \tag{5.7}$$

everywhere, in complete analogy with the quark constant (4.7) considered above. In this case from all the relations it follows that the gluon full self-energy coincides with its subtracted counterpart, and both quantities become purely transversal, since

$$\Pi(q^2; D^{PT}) = \Pi^s(q^2; D^{PT}), \quad \tilde{\Pi}(q^2; D^{PT}) = \tilde{\Pi}^s(q^2; D^{PT}) = 0. \tag{5.8}$$

The one way to satisfy the ST identity and thus to maintain the color gauge structure of QCD is to discard the mass scale parameter  $\Delta^2(D)$  from the very beginning, i.e., put it formally zero  $\Delta^2(D) = 0$  in all the equations, relations, etc. In this limit the initial gluon SD equation (5.3) is modified as follows:

$$D_{\mu\nu}^{PT}(q) = D_{\mu\nu}^0(q) + D_{\mu\rho}^0(q)iT_{\rho\sigma}(q)q^2\Pi^s(q^2; D^{PT})D_{\sigma\nu}^{PT}(q), \tag{5.9}$$

and the corresponding Lorentz structure which appears in (5.2) becomes

$$d^{PT}(q^2) = \frac{1}{1 + \Pi^s(q^2; D^{PT})}. \tag{5.10}$$

It is easy to see that the gluon SD equation (5.9) automatically satisfies the ST identity (5.1) now. Evidently, in the formal  $\Delta^2(D) = 0$  limit we denote  $D_{\mu\nu}(q)$  and  $d(q^2)$  as  $D_{\mu\nu}^{PT}(q)$  and  $d^{PT}(q^2)$ , respectively (for reason see Sect. 7).



As it follows from above, in the formal  $\Delta^2(D) = 0$  limit the full gluon self-energy and its subtracted counterpart (4.11) become purely transversal, i.e.,

$$q_\rho \Pi_{\rho\sigma}(q; D^{PT}) = q_\rho \Pi_{\rho\sigma}^s(q; D^{PT}) = 0. \tag{5.11}$$

Let us remind that the invariant function  $\Pi^s(q^2; D^{PT})$  can be only logarithmical divergent in the  $q^2 \rightarrow \infty$  limit. However, the formal  $\Delta^2(D) = 0$  limit implies  $\Delta_t^2(D) = \Delta_g^2(D) = 0$ , since they are independent quantities, see (4.10) and remarks after it. As we already know, in this case there will be no problems for ghosts to accomplish their role, namely to cancel the longitudinal component in the full gluon propagator (5.9). For the explicit cancellation in lower order of PT, see, for example, [2–4]. However, such a cancellation should occur in every order of PT, in accordance with the transversality condition (4.9).

We have reminded some important aspects of the color gauge structure of QCD, but without any use of PT.

### 5.1 A Preliminary Discussion

A few clarifying remarks are in order already at this stage. By substituting the initial gluon SD equation (2.1) into the ST identity, on account of the first of the general decompositions (4.11), one finally arrives at the conclusion that the full gluon self-energy should be transversal, as it follows from the condition (5.5). However, this is equivalent to the relation (5.6), which has only unique solution (5.7), namely put formally  $\Delta^2(D)$  zero everywhere. The full gluon self-energy (4.11) then becomes

$$\Pi_{\rho\sigma}(q; D^{PT}) = T_{\rho\sigma}(q)q^2\Pi(q^2; D^{PT}) = T_{\rho\sigma}(q)q^2\Pi^s(q^2; D^{PT}), \tag{5.12}$$

due to the relations (5.5), (5.7), (5.8) and (4.13). This means that if the gluon self-energy is transversal, then it cannot have the pole-type singularity, since  $\Pi(q^2; D^{PT}) = \Pi^s(q^2; D^{PT})$ , and  $\Pi^s(q^2; D^{PT})$  is always regular at  $q^2 \rightarrow 0$ . In other words, when  $\Delta^2(D) = 0$  everywhere, then from above it follows that  $\Pi_{\rho\sigma}(q; D^{PT}) = 0$ ,  $q^2 \rightarrow 0$ , and this is in complete agreement with  $\Pi_{\rho\sigma}(0; D^{PT}) = 0$ , which comes from the subtraction (3.1) at  $\Delta^2(D) = 0$ . It worth emphasizing once more that a massless single particle singularity appears only together with  $\Delta^2(D)$ . If it is zero then there is no singularity, and the gluon self-energy coincides with its subtracted counterpart, being both transversal (see the initial relations (5.11) and (5.12), which are valid only at zero  $\Delta^2(D)$ ). The situation is absolutely similar to the situation in QED (see Sect. 4.1).

On the other hand, let us substitute the general decomposition of the full gluon propagator (5.2) into the initial gluon SD equation (2.1), or, equivalently, (5.3), on account of the general relations (4.11) and (4.13), then one obtains

$$d(q^2) = \frac{1}{1 + \Pi^s(q^2; D) + (\Delta^2(D)/q^2)}, \tag{5.13}$$

and simultaneously the relation (5.5), namely

$$\tilde{\Pi}(q^2; D) = 0, \tag{5.14}$$

but which requires  $\Delta^2(D) = 0$  as it was repeatedly discussed above (see relations (5.5) and (5.7)). So, we need to disregard  $\Delta^2(D)$  term in the transversal component of the full gluon propagator, shown in (5.13), as well. In this way we are arriving at the system of (5.9)–(5.10). We call this system of equations as the PT one, since we have to put  $\Delta^2(D)$  zero everywhere, and replacing  $D \rightarrow D^{PT}$  in this limit.

However, just this was the first problem. Our aim is to retain  $\Delta^2(D)$  in the gluon SD equation (5.3), or equivalently, in the relation (5.13), and at the same time to automatically cancel  $\tilde{\Pi}(q^2; D)$  from (5.3), i.e., without appealing to the condition (5.14). That is why we need to introduce the spurious mechanism which makes it possible to achieve this goal (see especially (6.5) below, where  $\tilde{\Pi}(q^2; D)$  is not required to be zero, i.e., it remains arbitrary). This also makes it possible to satisfy the ST identity at non-zero  $\Delta^2(D)$ , since  $\tilde{\Pi}(q^2; D)$  is removed from the gluon SD equation by the spurious technics, and therefore there is no necessity to put it zero (see Sect. 6).

The second problem which we have met is that at non-zero  $\Delta^2(D)$  the transversality of the full gluon self-energy will be always violated (see the general relations (4.14) and (4.15)). In turn this means that the ghosts cannot cancel the longitudinal component of the full gluon propagator in this case. This problem occurs at any gluon momentum, and not only at zero gluon momentum  $\Pi_{\rho\sigma}(0; D) = \delta_{\rho\sigma} \Delta^2(D)$ , as it follows from the subtraction (3.1) at this point. So there is no solution of this problem at the level of the gluon self-energy. However, we have previously noticed that, in principle, we need rather the relevant gluon propagator to be transversal at non-zero  $\Delta^2(D)$  than the full gluon self-energy. Just how to avoid this difficulty (and thus to neutralize its negative consequences) for the relevant gluon propagator at non-zero  $\Delta^2(D)$ , i.e., to make it transversal without ghosts, will be explained and advocated in detail in Sect. 9.

## 6 The General Structure of the Full Gluon Propagator

The formal  $\Delta^2(D) = 0$  limit is a real way how to preserve the color gauge invariance in QCD. Then a natural question arises why does the mass scale parameter  $\Delta^2(D)$  (which is nothing but the re-defined tadpole term) exist in this theory at all? There is no doubt that the color gauge invariance of QCD should be maintained at non-zero mass scale parameter as well, since it is explicitly present in the full gluon self-energy, and hence in the full gluon propagator. However, by keeping it “alive”, the two important problems mentioned above arise. The first problem is how to replace the original gluon SD equation (5.3), since it is not consistent with the ST identity unless the mass scale parameter is discarded from the very beginning. The second problem is how to make the full gluon propagator purely transversal when the mass scale parameter is explicitly present.

### 6.1 The Spurious Mechanism

By introducing the spurious technics here, we will be able to show that the ST identity (5.1) can be automatically satisfied at non-zero  $\Delta^2(D)$  as well. As we already know, the original gluon SD equation (5.3) should be correspondingly modified in this case. In other words, our aim is to save  $\Delta^2(D)$  in the transversal invariant function (4.13), while removing it from the longitudinal invariant function (4.12), but without going to the formal  $\Delta^2(D) = 0$  limit, as it has been described in the previous section. In order to keep the mass scale parameter “alive”, and, at the same time, to satisfy the ST identity (5.1), it is instructive to introduce a

temporary dependence on  $\Delta^2(D)$  in the free gluon propagator, thus making it an auxiliary (spurious) free gluon propagator. The original gluon SD equation (5.3), then should read

$$D_{\mu\nu}(q) = D_{\mu\nu}^0(q; \Delta^2(D)) + D_{\mu\rho}^0(q; \Delta^2(D))iT_{\rho\sigma}(q)[q^2\Pi^s(q^2; D) + \Delta^2(D)]D_{\sigma\nu}(q) + D_{\mu\rho}^0(q; \Delta^2(D))iq^2\tilde{\Pi}(q^2; D)L_{\rho\sigma}(q)D_{\sigma\nu}(q) \tag{6.1}$$

with the spurious free gluon propagator as follows:

$$D_{\mu\nu}^0(q; \Delta^2(D)) = D_{\mu\nu}^0(q) + i\xi L_{\mu\nu}(q)d_0(q^2; \Delta^2(D))\frac{1}{q^2}. \tag{6.2}$$

We already know from the relation (5.4) that the spurious free gluon propagator  $D_{\mu\nu}^0(q; \Delta^2(D))$  can deviate from the standard free gluon propagator (2.2) only in its longitudinal component. The latter automatically satisfies the ST identity (5.1), while for the former this may not be true, indeed.

Substituting this sum into the gluon SD equation (6.1), one obtains

$$D_{\mu\nu}(q) = D_{\mu\nu}^0(q) + D_{\mu\rho}^0(q)iT_{\rho\sigma}(q)[q^2\Pi^s(q^2; D) + \Delta^2(D)]D_{\sigma\nu}(q) + I_{\mu\nu}(q; \Delta^2(D)), \tag{6.3}$$

where

$$I_{\mu\nu}(q; \Delta^2(D)) = i\xi d_0(q^2; \Delta^2(D))\left[L_{\mu\nu}(q) + L_{\mu\sigma}(q)iq^2\tilde{\Pi}(q^2; D)D_{\sigma\nu}(q)\right]\frac{1}{q^2} + D_{\mu\rho}^0(q)iq^2\tilde{\Pi}(q^2; D)L_{\rho\sigma}(q)D_{\sigma\nu}(q) = i\xi L_{\mu\nu}(q)\left[d_0(q^2; \Delta^2(D))\left(1 - \xi\tilde{\Pi}(q^2; D)\right) - \xi\tilde{\Pi}(q^2; D)\right]\frac{1}{q^2}. \tag{6.4}$$

Evidently, just this term violates the ST identity (5.1) in (6.3), so it should be zero, which implies

$$d_0(q^2; \Delta^2(D)) = \frac{\xi\tilde{\Pi}(q^2; D)}{1 - \xi\tilde{\Pi}(q^2; D)}. \tag{6.5}$$

Thus the gluon SD equation (6.3) finally becomes

$$D_{\mu\nu}(q) = D_{\mu\nu}^0(q) + D_{\mu\rho}^0(q)iT_{\rho\sigma}(q)[q^2\Pi^s(q^2; D) + \Delta^2(D)]D_{\sigma\nu}(q). \tag{6.6}$$

Evidently, from now on we can completely forget about the spurious free gluon propagator. It played its role and retired from the scene.

The modified gluon SD equation (6.6) is satisfied by the same expression for the Lorentz structure  $d(q^2)$  in (5.2) as the original gluon SD equation (5.3) shown in (5.13), namely

$$d(q^2) = \frac{1}{1 + \Pi^s(q^2; D) + (\Delta^2(D)/q^2)}, \tag{6.7}$$

which is not surprising, since the original gluon SD equation (5.3) and its modified version (6.6) differ from each other only by the longitudinal part.

However, the important observation is that now it is not required to put the mass scale parameter  $\Delta^2(D)$  formally zero everywhere. The spurious mechanism does not affect the dynamical context of the original gluon SD equation. In other words, it makes it possible

to retain the mass scale parameter in the transversal part of the gluon SD equation, and, at the same time, to cancel the term in its longitudinal part, which violates the ST identity. In this way, the modified gluon SD equation (6.6) satisfies automatically the ST identity. That is why we consider the modified gluon SD equation (6.6) as more general than its original counterpart (5.3).

The relation (6.7) cannot be considered as the formal solution for the full gluon propagator  $D$  in (5.2). The general mass scale term contribution  $(\Delta^2(D)/q^2)$  and the invariant function  $\Pi^s(q^2; D)$  themselves depend on  $D$ . In fact, it is a transcendental NL equation for determining  $d(q^2)$  as a function of  $\Delta^2(D)$  (see Sect. 8 below).

### 7 NP QCD vs. PT QCD

In the previous section it has been explicitly shown how the gluon SD equation should be modified in order to automatically satisfy the ST identity at non-zero mass scale parameter. It is instructive to collect our results here.

#### 7.1 NP QCD

The modified gluon SD equation (6.6) is

$$D_{\mu\nu}(q; \Delta^2(D)) = D_{\mu\nu}^0(q) + D_{\mu\rho}^0(q)iT_{\rho\sigma}(q)[q^2\Pi^s(q^2; D) + \Delta^2(D)]D_{\sigma\nu}(q; \Delta^2(D)), \tag{7.1}$$

while the general tensor decomposition is the standard one (5.2), namely

$$D_{\mu\nu}(q; \Delta^2(D)) = i \{T_{\mu\nu}(q)d(q^2; \Delta^2(D)) + \xi L_{\mu\nu}(q)\} \frac{1}{q^2}, \tag{7.2}$$

and the “solution” for its Lorentz structure is

$$d(q^2; \Delta^2(D)) = \frac{1}{1 + \Pi^s(q^2; D) + (\Delta^2(D)/q^2)}. \tag{7.3}$$

This system of equations forms the system of equations for NP QCD, since we distinguish between NP QCD and PT QCD by the explicit presence of the mass scale parameter, see discussion below (that is why we introduce temporarily the dependence on it in all the quantities above).

#### 7.2 PT QCD

The complete set of equations for PT QCD is

$$D_{\mu\nu}^{PT}(q) = D_{\mu\nu}^0(q) + D_{\mu\rho}^0(q)iT_{\rho\sigma}(q)q^2\Pi^s(q^2; D^{PT})D_{\sigma\nu}^{PT}(q), \tag{7.4}$$

with

$$D_{\mu\nu}^{PT}(q) = i \{T_{\mu\nu}(q)d^{PT}(q^2) + \xi L_{\mu\nu}(q)\} \frac{1}{q^2}, \tag{7.5}$$

and the “solution” for its Lorentz structure is

$$d^{PT}(q^2) = \frac{1}{1 + \Pi^s(q^2; D^{PT})}. \tag{7.6}$$

In both systems of equations the free gluon propagator is given in (2.2). The NP QCD system of equations has been obtained with the help of the spurious mechanism. It made it possible to keep the mass scale parameter  $\Delta^2(D)$  “alive”, and, at the same time, to automatically satisfy the ST identity. The PT QCD system of equations has been obtained by putting it formally zero everywhere. Evidently, the PT system of equations can be obtained from the NP system of equations in the formal  $\Delta^2(D) = 0$  limit, since the dependence of the latter system of equations on the mass scale parameter  $\Delta^2(D)$  is a regular one.

Due to asymptotic freedom (AF) in QCD the PT regime is realized at  $q^2 \rightarrow \infty$ . In this limit all the Green’s functions are possible to approximate by their free PT counterparts (up to the corresponding PT logarithms). However, from the relation (7.3) it follows that in this limit the mass scale term contribution  $\Delta^2(D)/q^2$  is only next-to-next-to-leading order one. The leading order contribution is the subtracted gluon self-energy  $\Pi^s(q^2; D)$ , which behaves like  $\ln q^2$  in this limit, as mentioned above. The constant 1 is the next-to-leading order term in the  $q^2 \rightarrow \infty$  limit. Such a special structure of the relation (7.3), namely the mass scale parameter enters it through the combination  $\Delta^2(D)/q^2$  in its denominator only, explains immediately why the mass scale parameter  $\Delta^2(D)$  is not important in PT. From this structure it follows that the PT regime at  $q^2 \rightarrow \infty$  is effectively equivalent to the formal  $\Delta^2(D) = 0$  limit and vice versa. That is the reason why this limit can be called the PT limit. And that is why we denote  $D_{\mu\nu}(q; \Delta^2 = 0) = D_{\mu\nu}(q; 0) \equiv D_{\mu\nu}^{PT}(q)$ , and hence  $d(q^2; \Delta^2 = 0) = d(q^2; 0) \equiv d^{PT}(q^2)$ , etc., in accordance with the previous notations.

Thus the formal PT  $\Delta^2(D) = 0$  limit exists, and it is a regular one. As it follows from above, in this limit one recovers the PT QCD system of equations from the NP QCD one. So, we distinguish between the PT and NP phases in QCD by the explicit presence of the mass scale parameter. Its aim is to be responsible for the NP QCD dynamics, since it dominates at  $q^2 \rightarrow 0$  in the “solution” (7.3). When it is put formally zero, then the PT phase survives only. Evidently, when such a scale is explicitly present then the QCD coupling constant plays no role in the NP QCD dynamics.

The mass scale parameter term does not survive in the PT  $q^2 \rightarrow \infty$  regime, anyway. Then it is justified to simply drop it in the PT. It is worth emphasizing that this does not depend on how it has been regularized. However, as underlined above, any regularization scheme should be adjusted to the ST identity (5.1). In fact, in the most popular dimensional regularization method (DRM) [6] it is prescribed to put  $\Delta^2(D_0) = 0$  (see also [2, 4] and especially the corresponding discussion in [3]). So it preserves the color gauge invariance in PT QCD from the very beginning.

## 8 The Mass Gap

One of the important challenges of QCD is that the Lagrangian of QCD [1–4] does not contain a mass scale parameter which could have a physical meaning even after the corresponding renormalization program is performed. The only place where it appears explicitly is the gluon SD equation of motion, as it has been described in this work, i.e., it is only due to the intrinsically NP (INP) dynamics of QCD developed in the gluon sector. This underlines the importance of the investigation of the SD system of equations and the corresponding ST identities ([1, 5, 7] and references therein) for understanding of the true dynamics in the QCD ground state. The propagation of gluons is one of the main dynamical effects there. The importance of the corresponding equation of motion is due to the fact that its solutions are supposed to reflect the quantum-dynamical structure of the QCD ground state (as mentioned above, this equation is highly NL, so the number of independent solutions is not

fixed *a priori*. From the very beginning they should be considered on equal footing). The color gauge structure of this equation has been investigated in this work.

In two-dimensional QCD the transversality condition (4.8) is satisfied, i.e., it is zero. This means that the tadpole term should be included from the very beginning. Otherwise, the ghosts will not be able to cancel the longitudinal component of the full gluon propagator [2]. However, this theory has already the scale parameter of dimension mass, which is the coupling constant. This once more emphasizes the special status of the tadpole term, and hence of the general mass scale parameter (4.10), in four-dimensional QCD.

The explicit presence of the general mass scale parameter (which is nothing but the re-defined tadpole term) in the full gluon propagator is no coincidence. On the one hand, it does not contradict the color gauge invariance of QCD. As it has been explicitly shown so far in this investigation it is compatible with the ST identity. On the other hand, it makes it possible to introduce the mass gap so needed in NP QCD in order to explain color confinement and other NP effects [8].

For further discussion it is convenient to re-write the relation (7.3) in the form of the corresponding transcendental equation, namely

$$d(q^2) = 1 - \left[ \Pi^s(q^2; d) + \frac{\Delta^2(d)}{q^2} \right] d(q^2), \tag{8.1}$$

where instead of  $D$  we introduced an equivalent dependence on  $d$ , i.e.,  $\Delta^2(D) \equiv \Delta^2(d)$ . The general mass scale parameter  $\Delta^2(d) \equiv \Delta^2(\lambda, \alpha, \xi, g^2; d)$ , where  $g^2$  is the dimensionless coupling constant squared, can be present as follows:

$$\Delta^2(\lambda, \alpha, \xi, g^2; d) = \Delta^2 \times c(\lambda, \alpha, \xi, g^2; d), \tag{8.2}$$

where the mass squared

$$\Delta^2 \equiv \Delta^2(\lambda, \alpha; \xi, g^2), \tag{8.3}$$

will be called the mass gap. Contrary to the arbitrary dimensionless constant  $c(\lambda, \alpha, \xi, g^2; d)$ , it does not depend on  $d$ , but may, in general, depend on  $\lambda, \alpha, \xi, g^2$ , and so on. Thus at this stage it is only regularized as well as the mass scale parameter itself.

If it will survive the renormalization program, then QCD is a complete and self-consistent theory without the need to introduce some extra degrees of freedom in order to generate a mass gap. We should prove that the product

$$\Delta_R^2 = Z(\lambda, \alpha, \xi, g^2) \Delta^2(\lambda, \alpha, \xi, g^2), \quad \lambda \rightarrow \infty, \alpha \rightarrow 0, \tag{8.4}$$

where  $Z(\lambda, \alpha, \xi, g^2)$  is the multiplicative renormalization (MR) constant of the mass gap, exists in the above-shown limits. However, the final result of these limits, i.e.,  $\Delta_R^2$ , should be achieved in the way not to compromise the general renormalizability of QCD. Contrary to the regularized version, the renormalized mass gap (8.4) should not depend on the gauge-fixing parameter, should be finite, positive definite, etc. Only after performing this program, one can assign to it a physical meaning of a scale responsible for the true NP dynamics of QCD at large distances, in the same way as  $\Lambda_{QCD}^2$  is responsible for its nontrivial PT dynamics at short distances. Apparently, the renormalized mass gap can be identified/related with/to the Jaffe and Witten mass gap [8], if it is possible at all. The MR program will be positively resolved in the subsequent papers. For this we have to explicitly find  $d(q^2)$  as a function of the regularized mass gap  $\Delta^2$  in the general way, and in particular with the help of (8.1).

Concluding, it is worth emphasizing that the formal PT  $\Delta^2(d) = 0$  limit implies to put the mass gap formally zero as well, namely  $\Delta^2 = 0$ . The arbitrary coefficient  $c(\lambda, \alpha, \xi, g^2; d)$ , which appears in (8.2), is, in general, not zero. In the rest of this paper instead of the general mass scale parameter  $\Delta^2(d)$  we will use the mass gap  $\Delta^2$  itself, for simplicity.

### 9 Restoration of Transversality of the Gluon Propagator in NP QCD

The NP QCD system of (7.1)–(7.3) depends explicitly on the mass gap  $\Delta^2$ . As we already know from above, then the ghosts are not able to cancel the longitudinal component in the full gluon propagator (7.2), i.e., they are of no use in this case (the transversality condition for the full gluon self-energy is always violated, see relations (4.14) and (4.15)). This is the price we have paid to keep the mass gap “alive” in the full gluon propagator. Our aim here is to formulate a method which allows one to make the gluon propagator, relevant for NP QCD, purely transversal in a gauge invariant way, even if the mass gap is explicitly present.

For this purpose let us define the truly NP (TNP) part of the full gluon propagator as follows:

$$D_{\mu\nu}^{TNP}(q; \Delta^2) = D_{\mu\nu}(q; \Delta^2) - D_{\mu\nu}(q; \Delta^2 = 0) = D_{\mu\nu}(q; \Delta^2) - D_{\mu\nu}^{PT}(q), \tag{9.1}$$

i.e., the subtraction is made with respect to the mass gap  $\Delta^2$ , and therefore the separation between these two terms is exact. Evidently, the formal PT  $\Delta^2(D) = 0$  limit is equivalently replaced by the formal mass gap limit to zero, i.e.,  $\Delta^2 = 0$ , as underlined above. So on account of the expressions (7.2) and (7.5), it becomes

$$D_{\mu\nu}^{TNP}(q; \Delta^2) = iT_{\mu\nu}(q) \left[ d(q^2; \Delta^2) - d^{PT}(q^2) \right] \frac{1}{q^2} = iT_{\mu\nu}(q) d^{TNP}(q^2; \Delta^2) \frac{1}{q^2}, \tag{9.2}$$

where the explicit expression for the TNP Lorentz structure  $d^{TNP}(q^2; \Delta^2)$  can be obtained from the relations (7.3) and (7.6) for  $d(q^2; \Delta^2)$  and  $d^{PT}(q^2)$ , respectively.

The subtraction (9.1) is equivalent to

$$D_{\mu\nu}(q; \Delta^2) = D_{\mu\nu}^{TNP}(q; \Delta^2) + D_{\mu\nu}^{PT}(q). \tag{9.3}$$

The TNP gluon propagator (9.2) does not survive in the formal PT  $\Delta^2 = 0$  limit. This means that it is free of the PT contributions, by construction. The full gluon propagator (7.2) in this limit is reduced to its PT counterpart (7.5). This means that the full gluon propagator, being also NP, nevertheless, is “contaminated” by them. The TNP gluon propagator is purely transversal in a gauge invariant way (no special (Landau) gauge choice by hand), while its full counterpart has a longitudinal component as well. There is no doubt that the true NP dynamics of the full gluon propagator is completely contained in its TNP part, since the subtraction (9.3) is nothing but adding zero to the full gluon propagator. We can write

$$\begin{aligned} D_{\mu\nu}(q; \Delta^2) &= i \{ T_{\mu\nu}(q) d(q^2; \Delta^2) + \xi L_{\mu\nu}(q) \} \frac{1}{q^2} - iT_{\mu\nu}(q) d^{PT}(q^2) \frac{1}{q^2} \\ &\quad + iT_{\mu\nu}(q) d^{PT}(q^2) \frac{1}{q^2} \\ &= D_{\mu\nu}^{TNP}(q; \Delta^2) + D_{\mu\nu}^{PT}(q), \end{aligned} \tag{9.4}$$

and so the true NP dynamics in the full gluon propagator is not affected, but contrary exactly separated from its PT dynamics, indeed. In other words, the TNP gluon propagator is the full gluon propagator but free of its PT “tail”.

### 9.1 Prescription

Taking this important observation into account, we propose instead of the full gluon propagator (7.2) to use its TNP counterpart (9.2) as the relevant gluon propagator for NP QCD, i.e., to replace

$$D_{\mu\nu}(q; \Delta^2) \rightarrow D_{\mu\nu}^{TNP}(q; \Delta^2) = D_{\mu\nu}(q; \Delta^2) - D_{\mu\nu}^{PT}(q), \tag{9.5}$$

and hence

$$d(q^2; \Delta^2) \rightarrow d^{TNP}(q^2; \Delta^2) = d(q^2; \Delta^2) - d^{PT}(q^2). \tag{9.6}$$

The subtraction (9.5) plays effectively the role of ghosts in our proposal (for its additional arguments and motivation see Appendix A). However, the ghosts cancel only the longitudinal component in the PT gluon propagator (7.5), while our proposal leads to the cancellation of the PT contribution in the full gluon propagator (7.2) as well (and thus to an automatical cancellation of its longitudinal component). Nevertheless, this is not a problem, since the mass gap is not survived in the formal PT limit, anyway.

In fact, our proposal is reduced to a rather simple prescription. If one knows a full gluon propagator, and is able to identify the mass scale parameter responsible for the NP dynamics in it, then the full gluon propagator should be replaced in accordance with the subtraction (9.5). The only problem with it is that, being exact, it may not be unique. However, the uniqueness of such kind of separation can be achieved only in the explicit solution for the full gluon propagator as a function of the mass gap (see the above-mentioned subsequent papers). Anyway, this subtraction is a first necessary step, which guarantees transversality of the TNP gluon propagator  $D_{\mu\nu}^{TNP}(q; \Delta^2)$  without losing even one bit of information on the true NP dynamics in the full gluon propagator  $D_{\mu\nu}(q; \Delta^2)$ . At the same time, its non-trivial PT dynamics is completely saved in its PT part  $D_{\mu\nu}^{PT}(q)$ . So it is worth emphasizing that the both terms in the subtraction (9.3) are valid in the whole momentum range, i.e., they are not asymptotics.

The full gluon propagator (7.2), keeping the mass gap “alive”, is not “physical” in the sense that it cannot be made transversal by ghosts. Therefore it cannot be used for numerical calculations of the physical observables from first principles. However, our proposal makes it possible to present it as the exact sum of the two “physical” propagators. The TNP gluon propagator is automatically transversal, by construction. It fully contains all the information of the full gluon propagator on its NP context. Just it should be used in accordance with the prescription (9.5) in order to calculate the physical observables in low-energy QCD.

On the other hand, in high-energy QCD the full gluon propagator should be replaced as follows:

$$D_{\mu\nu}(q; \Delta^2) \rightarrow D_{\mu\nu}^{PT}(q) = D_{\mu\nu}(q; \Delta^2) - D_{\mu\nu}^{TNP}(q; \Delta^2), \tag{9.7}$$

and hence

$$d(q^2; \Delta^2) \rightarrow d^{PT}(q^2) = d(q^2; \Delta^2) - d^{TNP}(q^2; \Delta^2). \tag{9.8}$$

The PT gluon propagator  $D_{\mu\nu}^{PT}(q)$  is free of the mass gap, and hence the ghosts will cancel its longitudinal component in accordance with the transversality relation (4.9), making it thus



transversal (“physical”). The PT gluon propagator  $D_{\mu\nu}^{PT}(q)$  fully contains all the information of the full gluon propagator on its non-trivial PT context (scale violation, AF [1–4]).

It is instructive to obtain (9.2) by the substitution into the subtraction (9.1) not the corresponding “solutions” but rather the corresponding modified gluon SD equation (7.1) itself and (7.4). Doing so, one obtains

$$D_{\mu\nu}^{TNP}(q; \Delta^2) = D_{\mu\rho}^0(q)iT_{\rho\sigma}(q) [q^2\Pi^s(q^2; D) + \Delta^2] D_{\sigma\nu}(q; \Delta^2) - D_{\mu\rho}^0(q)iT_{\rho\sigma}(q)q^2\Pi^s(q^2; D^{PT})D_{\sigma\nu}^{PT}(q). \tag{9.9}$$

Substituting further into this equation the expressions (2.2), (7.2) and (7.5) and after doing some tedious algebra, one arrives at

$$D_{\mu\nu}^{TNP}(q; \Delta^2) = -iT_{\mu\nu}(q) \left[ \Pi^s(q^2; D) + \frac{\Delta^2}{q^2} \right] d(q^2; \Delta^2) \frac{1}{q^2} + iT_{\mu\nu}(q)\Pi^s(q^2; D^{PT})d^{PT}(q^2) \frac{1}{q^2}. \tag{9.10}$$

From the relations (7.3) and (7.6) it follows that

$$\left[ \Pi^s(q^2; D) + \frac{\Delta^2}{q^2} \right] d(q^2; \Delta^2) = 1 - d(q^2; \Delta^2), \tag{9.11}$$

$$\Pi^s(q^2; D^{PT})d^{PT}(q^2) = 1 - d^{PT}(q^2),$$

and substituting them back into the previous equation, one again arrives at (9.2).

Let us show explicitly the SD equation for the TNP gluon propagator. From (9.9), on account of the subtraction (9.3), one gets

$$D_{\mu\nu}^{TNP}(q; \Delta^2) = D_{\mu\rho}^0(q)iT_{\rho\sigma}(q)[-q^2\Pi^s(q^2; D^{PT}) + q^2\Pi^s(q^2; D) + \Delta^2]D_{\sigma\nu}^{PT}(q) + D_{\mu\rho}^0(q)iT_{\rho\sigma}(q)[q^2\Pi^s(q^2; D) + \Delta^2]D_{\sigma\nu}^{TNP}(q; \Delta^2), \tag{9.12}$$

where  $D_{\sigma\nu}^{PT}(q)$  satisfies its own (7.4).

The remarkable feature of this equation is that, by switching interaction off (i.e., setting formally  $\Pi^s(q^2; D^{PT}) = \Pi^s(q^2; D) = \Delta^2 = 0$ ), it cannot be reduced to the free gluon propagator, like this occurs for the full gluon propagator (7.1) and its PT counterpart (7.4). In other words, in the truly NP QCD the gluon propagator is always “dressed”. And this brings in one more serious argument in favor of the above proposed subtractions of all the types of the PT contributions. The emission and absorption of the colored dressed gluons can be suppressed at large distance by the renormalization of the mass gap (see subsequent papers). At the same time, there exists no such mechanism to do the same with the colored free gluons in order to ensure their confinement. So the correct theory of low-energy QCD should exclude the free gluon propagator from its formalism. This just takes place in the truly NP QCD as a result of the subtraction of the PT gluon propagator, which always contains the free gluon propagator.

Concluding, the solution of the above-mentioned two problems how to preserve the color gauge invariance/symmetry in QCD at non-zero mass gap completes our investigation in this paper. This means that from now on we can forget the relations (4.14) and (4.15) at all, since there are no any more their negative consequences for the truly NP QCD. In this connection let us underline that the initial subtraction (3.1) has been done in a gauge invariant way (i.e.,

not in separate propagators, which enter the skeleton loop integrals, contributing to the full gluon self-energy).

## 10 General Discussion

The general scale parameter (4.10), having the dimensions of mass squared, or, equivalently, the mass gap  $\Delta^2$  is dynamically generated in the QCD gluon sector. It is mainly due to the non-linear interaction of massless gluon modes. It is defined as the value of the full gluon self-energy at some finite point. Thus it has not been introduced by hand, since it is hidden in the skeleton loop integrals, contributing to the full gluon self-energy. To make its existence perfectly clear just the definition of the subtracted gluon self-energy in (3.1) has been proposed.

All the quantities considered in this paper are necessarily regularized, as a first step. However, nothing depends in our approach on the specific regularization scheme, preserving or not gauge invariance. It is impossible to perform any concrete calculations of the regularized skeleton loop integrals, containing unknown, in general, the full propagators and vertices. No any truncations/approximations/assumptions (which means no use of PT), special gauge choice, etc., have been made for them. Only algebraic, i.e., exact derivations have been done in this paper.

The mass gap  $\Delta^2$  violates explicitly the ST identity for the full gluon propagator, which satisfies the corresponding equation of motion. Also, in its presence the ghosts are not able to cancel the longitudinal component in the full gluon propagator in order to guarantee unitarity of the  $S$ -matrix in this theory. So it should be disregarded on the general grounds, i.e., put formally zero everywhere. We have explicitly shown that this formal limit is equivalent to the PT  $q^2 \rightarrow \infty$  limit and vice versa, leading thus to the formulation of the system of equations for PT QCD.

In order to confirm that the color gauge invariance/symmetry of QCD is maintained in the explicit presence of the mass gap, we have introduced the spurious mechanism. It makes it possible to modify the original gluon SD equation for the full gluon propagator in a such way that makes the ST identity (5.1) automatically satisfied at non-zero  $\Delta^2$ . At the same time, the dynamical context of the modified gluon SD equation is not affected, i.e., it is the same as of the original gluon SD equation. The “solution” (7.3) depends regularly on the mass gap, and it has a correct PT  $\Delta^2 = 0$  limit, shown in the relation (7.6). From it clearly follows that the effect of the mass gap dominates at  $q^2 \rightarrow 0$  and strongly suppressed in the PT  $q^2 \rightarrow \infty$  regime, so it is justified to simply disregard it in PT.

The standard gluon SD equation (2.1) suffers from the overlapping UV divergences. Its counterpart, which is free of them [9] (and references therein), has a much more complicated structure than (2.1). However, we hope that using the same technics (or its a more sophisticated version) one can achieve the same conclusion, that’s the mass gap is consistent with the color gauge invariance/symmetry. Quite possible that there is no point in this investigation, since the mass gap does not survive in the PT  $q^2 \rightarrow \infty$  limit (see a brief discussion in [10] as well). Anyway, this investigation should be done elsewhere.

Our “solution” (7.3) depends explicitly on the mass gap. As underlined above, in this case the ghosts are of no use to cancel the longitudinal component in the full gluon propagator. However, we have formulated a general method which makes it possible to achieve transversality of the full gluon propagator, relevant for NP QCD, in a gauge invariant way. It is based on the exact subtraction of the PT contribution from the full gluon propagator. Such obtained the TNP gluon propagator is purely transversal, maintaining thus unitarity of the  $S$ -matrix within our approach. It completely reproduces the true NP structure of the full gluon

propagator, and, at the same time, is free of the PT “contaminations” at the fundamental gluon level.

As pointed out above, we need no ghosts to ensure the cancellation of the longitudinal component in the full gluon propagator. Nevertheless, this does not mean that we need no ghosts at all. We need them in other sectors of QCD, for example in the quark ST identity, which contains the so-called ghost-quark scattering kernel explicitly [1]. This kernel still makes an important contribution to the identity even if the relevant gluon propagator is transversal [11–15]. Do not mix the TNP gluon propagator (9.2) with the full gluon propagator (7.2) in the Landau gauge. The former is transversal by construction in a gauge invariant way. The latter one becomes transversal only by choosing Landau gauge by hand, i.e., not in a gauge-invariant way (let us remind that the ghosts cannot make it transversal).

In place of mass gap, any other mass scale parameter might serve. This could be introduced into the full gluon propagator by hand, as an ansatz, or arise as a result of some specific approximation/truncation made in the gluon SD equation itself and hence in its solution, etc. Its origin is irrelevant for our method, the only request is that the full gluon propagator should regularly depend on it. However, none of the truncations/approximations or ansatzs made or introduced in the framework of any approach should undermine the above-discussed general role of ghosts in PT QCD. Our method just guarantees this.

## 11 Conclusions

We have discussed some important issues of the color gauge invariance/symmetry of QCD without use of PT. The basic relation of our analysis is the subtraction (3.1), which clearly shows the NL dynamical origin of the mass gap in the gluon sector of QCD. It has the dimensions of mass squared. All the quantities are necessary regularized, and only algebraic derivations have been done with them. The mass gap violates the ST identity (5.1) for the full gluon propagator, which satisfies the corresponding equation of motion. It also prevents the ghosts to cancel the longitudinal component in the full gluon propagator. In order to maintain the color gauge invariance/symmetry it should be disregarded from the very beginning, i.e., put formally zero everywhere.

However, by introducing the initial subtraction (3.1) and the spurious technics, we have explicitly shown how to satisfy the ST identity at non-zero mass gap as well. Our approach makes it possible to retain it in the transversal part of the gluon SD equation, while canceling the term in its longitudinal part, which violates the ST identity. So the modified gluon SD equation automatically satisfies the ST identity for the full gluon propagator. At the same time, its dynamical context is not affected. Its “solution” (7.3) depends regularly on the mass gap term, and has a correct PT limit, i.e., the mass gap contribution does not survive in the PT  $q^2 \rightarrow \infty$  regime. At the same time, the mass gap contribution dominates the structure of the “solution” (7.3) in the NP  $q^2 \rightarrow 0$  limit.

We have also formulated a general method which allows to derive the gluon propagator relevant for NP QCD, the so-called TNP gluon propagator (9.2). It regularly depends on the mass gap in the way when it is put formally zero then it also vanishes. The basic element in this method is the subtraction (9.1). It exactly separates the TNP part from its PT counterpart in the full gluon propagator. The TNP gluon propagator is purely transversal in a gauge invariant way, maintaining thus unitarity of the  $S$ -matrix in NP QCD. It completely reproduces the true NP structure of the full gluon propagator, and, at the same time, is free of the PT contributions (“contaminations”). Thus it cannot be reduced to the free gluon propagator when the interaction is formally switched off. This is important to correctly understand

the confinement mechanism in QCD. As emphasized above, the emission and absorption of free gluons at large distances cannot be suppressed, in principle. The TNP QCD provides the solution of this fundamental problem, and therefore the proposed subtraction (9.1) seems to be necessary. It makes the relevant gluon propagator transversal and excludes the free gluons from the theory at the same time. That is why we have argued that just the TNP QCD should be used in order to correctly calculate the physical observables, processes, etc. in low-energy QCD.

Briefly, our approach to NP QCD is based on the initial definition (3.1) and the above-mentioned spurious (Sect. 6) and subtraction (Sect. 9) methods. It makes it possible to maintain the ST identity for the full gluon propagator, when the mass gap (or any other mass scale parameter) is explicitly present in it. It also makes it possible to neutralize the negative consequences of the violation of the transversality condition for the full gluon self-energy, when the mass gap is kept “alive”. All other ST identities are not affected, and the color currents are conserved as well as no particular gauge choice made.

Concluding, we arrived at a dilemma. To maintain the color gauge symmetry of QCD and unitarity of its  $S$ -matrix the mass gap should be dropped, coming thus to PT QCD. To keep the mass gap then one should proceed in accordance with our method. It allows one to maintain both the above-mentioned symmetry and unitarity in a gauge invariant way, coming thus to the truly NP QCD. We distinguish between them by the explicit presence of the mass gap and not by the strength of the coupling constant. It plays no role when the mass gap is kept “alive”. On the other hand, the difference between the truly NP and NP QCD is that the former vanishes in the PT  $\Delta^2 = 0$  limit, while the latter survives, and is to be reduced to PT QCD (we mean the corresponding equations of motion (7.1), (9.12) and (7.4), of course).

*The common belief (which comes from PT) that the mass gap (which is nothing but the re-defined tadpole term) contradicts the color gauge invariance/symmetry of QCD is false. This fundamental symmetry is maintained/preserved at non-zero mass gap as well.*

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## Appendix A: Motivation

An ultimate goal of any fundamental theory like QCD is to describe the physical observables, processes, etc., from first principles. It has been already achieved in PT QCD, which correctly describes the behavior of QCD in the high energy/momentum limit. To do the same in the low-energy/momentum region is a formidable task because of the color confinement phenomenon, the dynamical mechanism of which is not yet understood [16–19]. However, first what we need in order to accomplish the above-mentioned goal in low-energy QCD is to define correctly the relevant gluon propagator; it should be purely transversal and should reproduce the true large-scale structure of the QCD ground state.

In order to justify our proposal how to satisfy these conditions, let us discuss briefly one of the important characteristics of the QCD ground state—the Bag constant. It is just defined as the difference between the PT and NP vacuum energy densities (VED) [20–23]. So, we can symbolically put  $B = VED^{PT} - VED$ , where  $VED$  is the NP but “contaminated” by the PT contributions (i.e., this is a full  $VED$  like the full gluon propagator). At the same time, in accordance with our method we can continue as follows:  $B = VED^{PT} - VED = VED^{PT} - [VED - VED^{PT} + VED^{PT}] = VED^{PT} - [VED^{TNP} + VED^{PT}] = -VED^{TNP} > 0$ , since the  $VED$  is always negative. Thus the Bag constant is nothing but the TNP  $VED$ , apart from the sign, by definition, and thus is completely free of the PT “contaminations”.

An adequate formalism for the calculation of the Bag constant from first principles is the effective potential approach for composite operators [24, 25]. In the absence of external sources it is nothing but the VED. To leading order it gives the VED as a special function of the Lorentz structure in the full gluon propagator, which should be then integrated out over its momentum. Thus, in order to correctly calculate the Bag constant in accordance with the above-mentioned definition, it is necessary to replace the full Lorentz structure by its TNP counterpart due to the prescription (9.6) within our method. For how to correctly define and actually calculate the Bag constant from first principles, by making all the necessary subtractions at all levels, see [26, 27].

In turn, via the well known relations (see remarks below) the Bag constant is related to many other important NP quantities in QCD, such as the gluon and quark condensates, the topological susceptibility, etc., which are defined beyond PT only [20, 21, 27, 28]. This means that they are determined by such correlation functions from which all the types of the PT contributions should be, by definition, subtracted. It is worth emphasizing that such type of the subtractions are inevitable also for the sake of self-consistency. As mentioned above, in low-energy QCD there exist relations between different correlation functions, for example, the Witten-Veneziano (WV) and Gell-Mann-Oakes-Renner (GMOR) formulae. The former [29–33] relates the pion decay constant and the mass of the  $\eta'$  meson to the topological susceptibility. The latter [20, 21, 34] relates the chiral quark condensate to the pion decay constant and its mass. The famous trace anomaly relation (see, for example [20, 21, 31–33] and references therein) relates the Bag constant to the gluon and quark condensates [20, 21, 26]. Defining thus the Bag constant, the topological susceptibility and the gluon and quark condensates by the subtraction of all the types of the PT contributions, it would not be self-consistent to retain them in the correlation function, determining the pion decay constant, and in the expressions for the pion and  $\eta'$  meson masses.

A few additional remarks about the subtraction of the PT contributions are in order. Let us remind that in lattice QCD [35] such kind of an equivalent procedure also exists. In order to prepare an ensemble of lattice configurations for the calculation of any NP quantity or to investigate some NP phenomena, the excitations and fluctuations of gluon fields of the PT origin and magnitude should be “washed out” from the vacuum. This goal is usually achieved by using “Perfect Actions”, “cooling”, “cycling”, etc. (see [16, 17] and references therein). Evidently, this is rather similar to our method in continuous QCD.

In QCD sum rules approaching the deep IR region from above, the IR sensitive contributions were parameterized in terms of a few quantities (the gluon and quark condensates, etc.), while the direct access to NP effects was blocked by the IR divergences [20, 21, 36]. In the phenomenological estimate of the gluon condensate by this method the PT gluon propagator integrated out over the deep IR region (where it certainly fails) is to be dropped (see discussion given by Shifman in [17]. To drop the term in one side of the equation, relation, etc. is equivalent to subtract the same term from its another side). The necessity of the subtraction of the PT part of the “running” coupling constant (integrated out) for the analytical calculation of the gluon condensate has been explicitly shown in [37].

In order to calculate correctly any truly NP quantity in low-energy QCD from first principles one has to begin with making subtractions at the fundamental quark-gluon level, anyway. The necessity of the subtractions at all levels is discussed in [10] in more detail. Our proposal (9.5)–(9.6) ensures transversality of the TNP gluon propagator in a gauge invariant way. This makes it possible to maintain unitarity of the  $S$ -matrix, even if the mass gap (or any other mass scale parameter) is explicitly present. The TNP gluon propagator correctly reproduces the true NP structure of the full gluon propagator, while being free of its PT contributions. This also excludes the free gluons from the theory, which, as underlined above, is important for the correct understanding of the color confinement mechanism.

## References

1. Marciano, W., Pagels, H.: Phys. Rep. C **36**, 137 (1978)
2. Peskin, M.E., Schroeder, D.V.: An Introduction to Quantum Field Theory. Addison-Wesley, Reading (1995)
3. Itzykson, C., Zuber, J.-B.: Quantum Field Theory. McGraw-Hill, New York (1984)
4. Muta, T.: Foundations of QCD. World Scientific, Singapore (1987)
5. Gogohia, V.: Phys. Lett. B **618**, 103 (2005)
6. 't Hooft, G., Veltman, M.: Nucl. Phys. B **44**, 189 (1972)
7. Alkofer, R., von Smekal, L.: Phys. Rep. **353**, 281 (2001)
8. Jaffe, A., Witten, E.: Yang-Mills Existence and Mass Gap, <http://www.claymath.org/prize-problems/>, <http://www.arthurjaffe.com>
9. Baker, M., Lee, Ch.: Phys. Rev. D **15**, 2201 (1977)
10. Gogokhia, V.: [arXiv:hep-ph/0606010](https://arxiv.org/abs/hep-ph/0606010)
11. Taylor, J.C.: Nucl. Phys. B **33**, 436 (1971)
12. 't Hooft, G.: Nucl. Phys. B **33**, 173 (1971)
13. Pagels, H.: Phys. Rev. D **15**, 2991 (1977)
14. Gogokhia, V.Sh.: Phys. Rev. D **40**, 4157 (1989)
15. Gogokhia, V.Sh.: Phys. Rev. D **41**, 3279 (1990)
16. van Baal, P.: Confinement, Duality, and Nonperturbative Aspects of QCD, NATO ASI Series B: Physics, vol. 368. Plenum, New York (1997)
17. Aoki, K.-I., Miyamura, O., Suzuki, T. (eds.): Non-perturbative QCD, structure of the QCD vacuum. Prog. Theor. Phys. Suppl. **131**, 1 (1998)
18. Gribov, V.N.: Gauge Theories and Quark Confinement. PHASIS, Moscow (2002)
19. Dokshitzer, Y.L., Kharzeev, D.E.: [hep-ph/0404216](https://arxiv.org/abs/hep-ph/0404216)
20. Shifman, M.A., Vainshtein, A.I., Zakharov, V.I.: Nucl. Phys. B **147**, 385, 448 (1979)
21. Novikov, V.A., Shifman, M.A., Vainshtein, A.I., Zakharov, V.I.: Nucl. Phys. B **191**, 301 (1981)
22. Shuryak, E.V.: Phys. Rep. **115**, 151 (1984)
23. Chanowitz, M.S., Sharpe, S.: Nucl. Phys. B **222**, 211 (1983)
24. Cornwall, J.M., Jackiw, R., Tomboulis, E.: Phys. Rev. D **10**, 2428 (1974)
25. Barducci, A., et al.: Phys. Rev. D **38**, 238 (1988)
26. Barnaföldi, G.G., Gogokhia, V.: [arXiv:0708.0163](https://arxiv.org/abs/0708.0163) [hep-ph]
27. Gogohia, V., Kluge, Gy.: Phys. Rev. D **62**, 076008 (2000)
28. Halperin, I., Zhitnitsky, A.: Nucl. Phys. B **539**, 166 (1999)
29. Witten, E.: Nucl. Phys. B **156**, 269 (1979)
30. Veneziano, G.: Nucl. Phys. B **159**, 213 (1979)
31. Gogohia, V.: Phys. Lett. B **501**, 60 (2001)
32. Gogohia, V., Toki, H.: Phys. Rev. D **61**, 036006 (2000)
33. Gogohia, V., Toki, H.: Phys. Rev. D **63**, 079901 (2001)
34. Gell-Mann, M., Oakes, R.J., Renner, B.: Phys. Rev. **175**, 2195 (1968)
35. Kronfeld, A.S.: [hep-ph/0209321](https://arxiv.org/abs/hep-ph/0209321)
36. Zakharov, V.I.: Int. J. Mod. Phys. A **14**, 4865 (1999)
37. Alekseev, A.I., Arbuzov, B.A.: [hep-ph/0407056](https://arxiv.org/abs/hep-ph/0407056), [hep-ph/0411339](https://arxiv.org/abs/hep-ph/0411339)